Q-deformation of Lie superalgebras B(m, n), B(0, n), C(1+n) and D(m, n) in their boson-fermion representations

1991 J. Phys. A: Math. Gen. 24 5451

(http://iopscience.iop.org/0305-4470/24/23/012)

J. Phys. A: Math. Gen. 24 (1991) 5451–5463. Printed in the UK

Q-deformation of Lie superalgebras $B(m,n)$, $B(0,n)$, $C(1+n)$ and $D(m,n)$ in their boson–fermion representations*

Li Liao† and Xing-Chang Song†‡

† CCAST (World Laboratory), PO Box 8730, Beijing 100080, and Department of Physics§, Peking University, Beijing 100871, People's Republic of China

‡ Institute of Theoretical Physics, Academia Sinica, Beijing 100080, People's Republic of China

Received 18 February 1991

Abstract. In this paper a systematical approach is proposed to give the q-deformation of basic Lie superalgebra (BLS) $B(m,n)$, $B(0,n)$, $C(1+n)$ and $D(m,n)$ in their boson–fermion oscillator representation with special attention paid to the $B(m,n)$ series.

1. Introduction

Since the proposal of a q-deformed oscillator was made, some papers using it to get q-deformation of Lie algebras and Lie superalgebras have appeared [1–3]. Among them is the more suggestive one, given in [4], in which the q-oscillator is constructed from the ordinary one and therefore the possibility of using it to quantize all algebras is implied.

With the active interest in supermathematics, especially in the theory of Lie superalgebras, it is natural to develop the q-deformation of Lie superalgebras despite a direct physical application of this q-deformation being still absent. In this paper, by developing a so-called graded q-analogue of Clifford algebra, we construct a q-deformation of basic Lie superalgebra (BLS) $B(m,n)$, $B(0,n)$, $C(1+n)$ and $D(m,n)$. We briefly recall the main properties of BLS in section 2 from the point of view of the Cartan matrix and Kac–Dynkin diagram. Section 3 sketches the main approach we use to obtain the q-deformed Lie superalgebra by analysing some concrete examples. Section 4 contributes to the boson–fermion representations of the BLS B, C, D in the general case, and the q-deformation of these algebras are discussed in section 5.

2. The main properties of BSL

The Lie bracket in a Lie superalgebra $g = g_0 + g_1$ is defined as

$$\langle a, b\rangle = ab - (-1)^{\deg(a)\deg(b)} ba \qquad \text{for all } a, b \in g \tag{2.1}$$

* This work is supported in part by the National Natural Science Foundation of China and the Doctoral Programme Foundation of the Institute of High Education.

§ Address for correspondence.

with the degree being zero for elements in the subalgebra g_0 (even part) and 1 for those in the g_0-representation g_1 (odd part). If dim $g_0 = n$ and dim $g_1 = m$, any element $a \in g$ can then be seen as an $(n+m)\times(n+m)$ matrix M:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.2}$$

where A (respectively D) is an $n\times n$ (respectively $m\times m$) matrix, and the supertrace of M is defined as

$$S\,\mathrm{Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D = \mathrm{Tr}\, M\eta \tag{2.3}$$

where Tr is the ordinary trace and η is similar to M with $B = C = 0$, $A = I_n$, $D = -I_m$.

The classification of simple Lie superalgebras is given by Kac [5]. Now we just write here those relevant basic superalgebras with Δ_0 (Δ_1) denoting the set of even (odd) roots. For the orthosymplectic series Osp($M|2n$), their even part g_0 is a non-compact form of $\mathrm{o}(M)\oplus\mathrm{sp}(2n)$, and their odd part g_1 (for $M\neq 2$) spans the $(M, 2n)$ g_0-representation. With the help of the fermionic parameter ε_i ($i = 1, 2, \ldots, m$) and the bosonic parameter δ_k ($k = 1, 2, \ldots, n$) satisfying

$$(\varepsilon_i, \varepsilon_j) = -\delta_{ij} \qquad (\delta_k, \delta_l) = \delta_{kl} \qquad (\varepsilon_i, \delta_k) = 0 \tag{2.4}$$

the roots can be expressed in the following way: for $B(m, n)$ or Osp($2m+1|2n$) with $m\neq 0$:

$$\begin{aligned} &\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j;\ \pm\varepsilon_i;\ \pm\delta_i \pm \delta_j;\ \pm 2\delta_i\} \qquad (i\neq j) \\ &\Delta_1 = \{\pm\delta_i;\ \pm\varepsilon_i \pm \delta_j\} \end{aligned} \tag{2.5}$$

for $B(0, n)$ or Osp($1|2n$):

$$\begin{aligned} &\Delta_0 = \{\pm\delta_i \pm \delta_j;\ \pm 2\delta_i\} \qquad (i\neq j) \\ &\Delta_1 = \{\pm\delta_i\} \end{aligned} \tag{2.6}$$

for $D(m, n)$ or Osp($2m|2n$) with $m\neq 1$:

$$\begin{aligned} &\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j;\ \pm\delta_i \pm \delta_j;\ \pm 2\delta_i\} \qquad (i\neq j) \\ &\Delta_1 = \{\pm\varepsilon_i \pm \delta_j\} \end{aligned} \tag{2.7}$$

while for $C(1+n)$ or Osp($2|2n$) the odd part g_1 is twice the fundamental $(2n)$ representation of sp($2n$), and the roots in terms of $\varepsilon, \delta_1, \ldots, \delta_n$ are

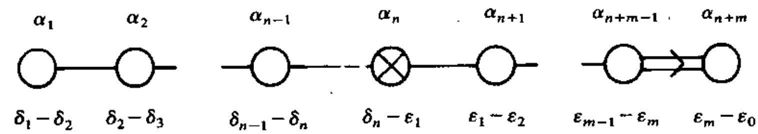
$$\begin{aligned} &\Delta_0 = \{\pm\delta_i \pm \delta_j;\ \pm 2\delta_i\} \qquad (i\neq j) \\ &\Delta_1 = \{\pm\varepsilon \pm \delta_i\}. \end{aligned} \tag{2.8}$$

For a BLS g of rank r, it is always possible to define an $r\times r$ Cartan matrix $A = (a_{ij})$ associated with a set of simple roots $(\alpha_1, \ldots, \alpha_r)$ with the following relations:

$$\begin{aligned} &[h_i, h_j] = 0 \\ &[h_i, e_{\pm\alpha_j}] = \pm a_{ij} e_{\pm\alpha_j} \\ &\langle e_{\alpha_i}, e_{-\alpha_j}\rangle = \delta_{ij} h_i \end{aligned} \tag{2.9}$$

where $h_1, \ldots, h_r$ generate the corresponding Cartan subalgebra H. Once again the so-called Kac–Dynkin diagram will be helpful, but a significant difference from the Lie algebras takes place at this point due to the unavoidable presence of odd roots together with even ones in the simple root systems. As a result, to a given superalgebra g will not in general be associated only one system of simple roots up to a transformation of the Weyl group, and therefore not only one Kac–Dynkin diagram. Here, a special simple root system with the characteristic of just containing the smallest number of odd roots is favoured [5, 6]:

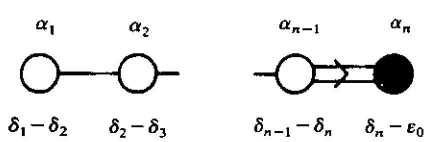
(i) $B(m,n)$, $(m>0)$



and the corresponding Cartan matrix

$$A=\begin{pmatrix}
2 & -1 & 0 & & & 0 & \cdot & \cdot & & \cdot & \cdot & 0\\
-1 & 2 & -1 & & & & & & & & & \cdot\\
0 & -1 & \cdot & \cdot & & & & & & & & \cdot\\
\cdot & & \cdot & \cdot & & & & & & & & \\
 & & & & & & & & & & & \cdot\\
\cdot & & & & 2 & -1 & & & & & & \cdot\\
0 & & & & -1 & 0 & 1 & & & & & 0\\
\cdot & & & & & -1 & 2 & -1 & & & & \cdot\\
\cdot & & & & & & & & & & & \\
 & & & & & & & \cdot & & & & \\
 & & & & & & & & \cdot & \cdot & & \cdot\\
\cdot & & & & & & & & \cdot & \cdot & -1 & 0\\
\cdot & & & & & & & & & -1 & 2 & -1\\
0 & \cdot & \cdot & \cdot & \cdot & 0 & & & & 0 & -2 & 2
\end{pmatrix}. \tag{2.10}$$

(ii) $B(0,n)$



and the corresponding Cartan matrix

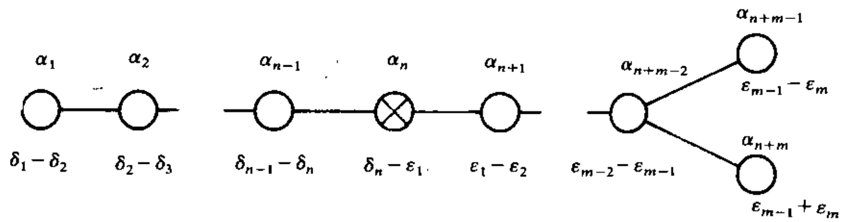
$$A=\begin{pmatrix}
2 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & & 0\\
-1 & 2 & -1 & & & & & & \cdot\\
0 & -1 & \cdot & \cdot & & & & & \cdot\\
\cdot & & \cdot & \cdot & \cdot & & & & \cdot\\
\cdot & & & \cdot & \cdot & \cdot & & & \cdot\\
\cdot & & & & \cdot & \cdot & \cdot & & \cdot\\
\cdot & & & & & \cdot & \cdot & -1 & 0\\
\cdot & & & & & & -1 & 2 & -1\\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -2 & 2
\end{pmatrix}. \tag{2.11}$$

Here for both the $B(m, n)$ and $B(0, n)$ cases we have introduced an auxiliary fermionic null vector parameter ε_0 with

$$(\varepsilon_0, \varepsilon_0) = 0 = (\varepsilon_0, \varepsilon_i) = (\varepsilon_0, \delta_k). \tag{2.12}$$

Evidently, it has no essential influence on parameter calculation and is introduced just for convenience at this step: roots given by difference of two bosonic vectors (δ, δ) or fermionic vectors $(\varepsilon, \varepsilon)$ are even, while roots given by difference of one bosonic and one fermionic vector are odd. However, in the following section we will see that ε_0 is deeply related to some feature of $o(2m+1)$.

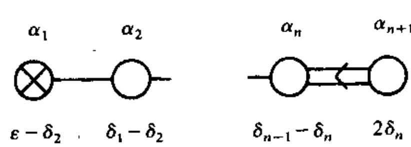
(iii) $D(m, n)$



and the corresponding Cartan matrix

$$A = \begin{pmatrix}
2 & -1 & 0 & & & & 0 & & & & & & 0 \\
-1 & 2 & -1 & & & & & & & & & & \\
0 & -1 & \cdot & \cdot & & & & & & & & & \\
 & & \cdot & \cdot & \cdot & & & & & & & & \\
 & & & \cdot & \cdot & \cdot & & & & & & & \\
 & & & & \cdot & \cdot & \cdot & & & & & & \\
 & & & & & 2 & -1 & & & & & & \\
0 & & & & & -1 & 0 & 1 & & & & & 0 \\
 & & & & & & -1 & 2 & -1 & & & & \\
 & & & & & & & \cdot & \cdot & \cdot & & & \\
 & & & & & & & & \cdot & \cdot & \cdot & & \\
 & & & & & & & & & \cdot & \cdot & \cdot & 0 \\
 & & & & & & & & & & \cdot & -1 & -1 \\
 & & & & & & & & & & -1 & 2 & 0 \\
0 & & & & & & 0 & & & 0 & -1 & 0 & 2
\end{pmatrix}. \tag{2.13}$$

(iv) $C(1+n)$



and the corresponding Cartan matrix

$$A=\begin{pmatrix} 0 & -1 & 0 & & & & & & 0 \\ -1 & 2 & -1 & & & & & & \cdot \\ 0 & -1 & \cdot & \cdot & & & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot & -1 & 0 \\ \cdot & & & & & & -1 & 2 & -2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}. \tag{2.14}$$

In the above, an open circle ○ denotes a simple even root, an open circle with a dot ⊙ the simple odd root α_i with $a_{ii}\neq 0$ and an open circle with a cross ⊗ the odd root with $a_{ii}=0$.

3. Examples

In this section we will consider some concrete examples to illustrate how to bring the orthosymplectic superalgebras into the oscillator form and then to deform them into their quantum version.

3.1. Osp(1/2) = B(0, 1)

For the simplest rank-one orthosymplectic superalgebra Osp(1|2) [3], only one bosonic parameter δ is needed. Another fermionic null vector parameter ε_0 is also introduced for convenience:

$$(\delta,\delta)=1 \qquad (\varepsilon_0,\varepsilon_0)=0=(\delta,\varepsilon_0).$$

Besides one zero root we have two bosonic roots,

$$\beta_\pm=\pm 2\delta \qquad (\beta_+,\beta_+)=4 \tag{3.1a}$$

together with two fermionic roots,

$$\alpha_\pm=\pm(\delta-\varepsilon_0) \qquad (\alpha_+,\alpha_+)=1. \tag{3.1b}$$

Correspondingly, we have five generators: one h from Cartan subalgebra and four $e_{\pm\alpha}$, $e_{\pm\beta}$ from non-zero roots. Three even generators form an angular momentum $\boldsymbol{J}$ (describing sp(2)) and two odd generators form $\frac{1}{2}$-rank irreducible tensor operators of $\boldsymbol{J}$:

$$h=4J_0 \qquad e_{\pm\beta}=\sqrt{2}J_\pm \qquad e_{\pm\alpha}=\sqrt{2}V_\pm. \tag{3.2}$$

They satisfy the following commutation and anticommutation relations:

$$[J_0,J_\pm]=\pm J_\pm \qquad [J_+,J_-]=2J_0 \tag{3.3a}$$

$$[J_0,V_s]=sV_s \qquad [J_\pm,V_s]=\sqrt{(\tfrac{1}{2}\mp s)(\tfrac{1}{2}\pm s+1)}\,V_{s\pm1} \qquad (s=\pm\tfrac{1}{2}) \tag{3.3b}$$

$$\{V_\pm,V_\pm\}=\mp 2J_\pm \qquad \{V_+,V_-\}=2J_0. \tag{3.3c}$$

Introducing the classical bosonic oscillators b, b^+ with $[b,b^+]=1$ and another auxiliary fermionic operator $a_0^+=a_0$, $a_0^2=1$, we can make the identifications

$$\begin{aligned} &J_+=-\tfrac{1}{2}b^+b^+ \qquad J_0=\tfrac{1}{2}(b^+b-\tfrac{1}{2}) \qquad J_-=\tfrac{1}{2}bb \\ &V_+=\frac{1}{\sqrt{2}}b^+a_0 \qquad V_-=\frac{1}{\sqrt{2}}ba_0 \end{aligned} \tag{3.4}$$

to reproduce the relations in equations (3.3). For the time being a_0 is only introduced for convenience, to indicate that $V_\pm$ are fermionic operators.

The simple root is the fermionic one α_+, and the whole $B(0,1)$ algebra can be generated by h and $e_{\pm\alpha}$, with

$$h = 2N+1 \qquad e_\alpha = b^+ a_0 \qquad e_{-\alpha} = a_0 b \tag{3.5}$$

where N is the number operator $N = b^+ b$. We see immediately that

$$[h, e_{\pm\alpha}] = \pm 2 e_{\pm\alpha} \qquad \{e_\alpha, e_{-\alpha}\} = h. \tag{3.6}$$

To pass over to the quantum enveloping algebra, one needs the q-deformed oscillator operators

$$\tilde{b}^+ \tilde{b} = [N] \qquad \tilde{b}\tilde{b}^+ = [1+N] \tag{3.7}$$

and choose

$$\tilde{e}_\alpha = \tilde{b}^+ a_0 \qquad \tilde{e}_{-\alpha} = a_0 \tilde{b}. \tag{3.8}$$

Then one gets

$$[h, \tilde{e}_{\pm\alpha}] = \pm 2 \tilde{e}_{\pm\alpha} \qquad \{\tilde{e}_\alpha, \tilde{e}_{-\alpha}\} = [N+\tfrac{1}{2}]/[\tfrac{1}{2}] = [h]_{q^{1/2}} \tag{3.9}$$

just as expected, since the root α is shorter compared with other roots in other algebras (cf the example in section 3.3). A similar result has also been reported by Chaichian *et al* [3].

3.2. $Osp(2/2) = C(1+1)$

This superalgebra [3] has rank 2, and is isomorphic to $A(1,0)$. Two parameters, one bosonic (δ) and one fermionic (ε), are introduced with $(\delta,\delta) = 1 = -(\varepsilon,\varepsilon)$, $(\delta,\varepsilon) = 0$. Two simple roots are chosen to be

$$\alpha_1 = \varepsilon - \delta \qquad \alpha_2 = 2\delta \tag{3.10}$$

with the first one fermionic and the second one bosonic. Another positive root is also fermionic, $\alpha_1 + \alpha_2 = \varepsilon + \delta$. Of the four even generators, a triplet forms an angular momentum J describing sp(2) and the fourth L generates $o(2) \approx gl(1)$. The odd part is constructed as the (2, 2) representation of $o(2) \oplus sp(2)$, which reduces to two sets of spinor operators of J, $V_s^{(r)}$, $r = \pm$, $s = \pm\frac{1}{2} \sim \pm$. The fundamental definition relations are given as follows:

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \qquad [L, J] = 0 \tag{3.11a}$$

$$\begin{aligned} &[L, V_s^{(r)}] = r V_s^{(r)} \qquad [J_0, V_s^{(r)}] = s V_s^{(r)} \\ &[J_\pm, V_s^{(r)}] = \sqrt{(\tfrac{1}{2} \mp s)(\tfrac{1}{2} \pm s + 1)}\, V_{s\pm 1}^{(r)} \end{aligned} \tag{3.11b}$$

$$\begin{aligned} &\{V_s^{(r)}, V_{s'}^{(r)}\} = 0 \\ &\{V_+^{(+)}, V_+^{(-)}\} = -2J_+ \qquad \{V_-^{(+)}, V_-^{(-)}\} = 2J_- \\ &\{V_+^{(+)}, V_-^{(-)}\} = 2J_0 - (L - \tfrac{1}{2}) \qquad \{V_-^{(+)}, V_+^{(-)}\} = 2J_0 + (L - \tfrac{1}{2}). \end{aligned} \tag{3.11c}$$

This algebra can be put into the oscillator form by introducing a pair of bosonic operators, $[b, b^+] = 1$, corresponding to the parameter δ, and a pair of fermionic operators, $\{a, a^+\} = 1$, $a^2 = a^{+2} = 0$, corresponding to ε. Then the identification

$$\begin{aligned} &J_+ = -\tfrac{1}{2} b^+ b^+ \qquad 2J_0 = N + \tfrac{1}{2} = b^+ b + \tfrac{1}{2} \qquad J_- = \tfrac{1}{2} bb \qquad L = a^+ a = M \\ &V_+^{(+)} = b^+ a^+ \qquad V_-^{(+)} = b a^+ \qquad V_+^{(-)} = a b^+ \qquad V_-^{(-)} = ab \end{aligned} \tag{3.12}$$

will reproduce the relations (3.11). Again, when we replace the operators a, b in the step generator with the deformed ones $\tilde{a}$, $\tilde{b}$,

$$\begin{aligned} &\tilde{a}^2=0=(\tilde{a}^+)^2 \qquad \tilde{a}^+\tilde{a}=[M] \qquad \tilde{a}\tilde{a}^+=[1-M] \\ &\tilde{b}^+\tilde{b}=[N] \qquad \tilde{b}\tilde{b}^+=[1+N] \end{aligned} \tag{3.13}$$

and keep the Cartan generators unchanged, we will get the q-deformation of the superalgebra $C(1+1)$. Then the anticommutations (3.11c) now take the form

$$\begin{aligned} \{\tilde{V}_+^{(+)}, \tilde{V}_-^{(-)}\} &= \tilde{b}^+\tilde{a}^+\tilde{b}\tilde{a}+\tilde{b}\tilde{a}\tilde{b}^+\tilde{a}^+=[N][M]+[1+N][1-M] \\ &=[N+1-M]=[2J_0-(L-\tfrac{1}{2})] \end{aligned} \tag{3.14a}$$

$$\begin{aligned} \{\tilde{V}_-^{(+)}, \tilde{V}_+^{(-)}\} &= \tilde{b}\tilde{a}^+\tilde{b}^+\tilde{a}+\tilde{b}^+\tilde{a}\tilde{b}\tilde{a}^+=[1+N][M]+[N][1-M] \\ &=[N+M]=[2J_0+(L-\tfrac{1}{2})] \end{aligned} \tag{3.14b}$$

and

$$\begin{aligned} \{\tilde{V}_+^{(+)}, \tilde{V}_+^{(-)}\}_q &= \tilde{b}^+\tilde{a}^+\tilde{b}^+\tilde{a}+q\tilde{b}^+\tilde{a}\tilde{b}^+\tilde{a}^+ \\ &=\tilde{b}^+\tilde{b}^+([M]+q[1-M]) \\ &=\tilde{b}^+\tilde{b}^+q^{1-M}=-[2]q^{1-M}\tilde{J}_+ \end{aligned} \tag{3.14c}$$

$$\{\tilde{V}_-^{(+)}, \tilde{V}_-^{(-)}\}_q=\tilde{b}\tilde{b}q^{1-M}=[2]q^{1-M}\tilde{J}_-. \tag{3.14d}$$

Here

$$\tilde{J}_+=-\frac{1}{[2]}\tilde{b}^+\tilde{b}^+ \qquad \tilde{J}_-=\frac{1}{[2]}\tilde{b}\tilde{b}. \tag{3.15}$$

Therefore

$$[\tilde{J}_+, \tilde{J}_-]=\frac{1}{[2]^2}\{[1+N][2+N]-[N][N-1]\}=\frac{[2N+1]}{[2]}=\frac{[4J_0]}{[2]}. \tag{3.16}$$

Other commutators in equations (3.11a) and (3.11b) remain unchanged.

In the standard notation we define the generators corresponding to the simple roots as

$$\begin{aligned} &e_{\alpha_1}=a^+b=V_-^{(+)} \qquad e_{-\alpha_1}=b^+a=V_+^{(-)} \qquad h_1=M+N=2J_0+(L-\tfrac{1}{2}) \\ &e_{\alpha_2}=-\tfrac{1}{2}b^+b^+=J_+ \qquad e_{-\alpha_2}=\tfrac{1}{2}bb=J_- \qquad h_2=N+\tfrac{1}{2}=2J_0. \end{aligned} \tag{3.17}$$

Then the fundamental commutation of Osp(2|2) can be written as

$$\begin{aligned} &[h_i, e_{\alpha_j}]=a_{ij}e_{\alpha_j} \qquad [h_i, e_{-\alpha_j}]=-a_{ij}e_{-\alpha_j} \\ &\langle e_{\alpha_i}, e_{-\alpha_j}\rangle=\delta_{ij}h_j \end{aligned} \tag{3.18}$$

where a_{ij} is the Cartan matrix, which for Osp(2|2) has the form

$$A=\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}.$$

For the quantum case $U_q(\mathrm{Osp}(2|2))$ we have

$$\begin{aligned} &[h_i, \tilde{e}_{\alpha_j}]=a_{ij}\tilde{e}_{\alpha_j} \qquad [h_i, \tilde{e}_{-\alpha_j}]=-a_{ij}\tilde{e}_{-\alpha_j} \\ &\{\tilde{e}_{\alpha_1}, \tilde{e}_{-\alpha_1}\}=[h_1] \qquad [\tilde{e}_{\alpha_2}, \tilde{e}_{-\alpha_2}]=[2h_2]/[2]\equiv[h_2]_{q^2}. \end{aligned} \tag{3.19}$$

The subscript q^2 in $[h_2]_{q^2}$ is an indication of the fact that the length square of the second root α_2 is twice that of the first one, α_1. Some of these results are also given by Deguchi *et al* [3].

3.3. $Osp(3|2) = B(1, 1)$

Now we discuss another rank-2 orthosymplectic algebra, Osp(3|2). The Kac-Dynkin diagram and Cartan matrix of $B(1, 1)$ are

$$\underset{\delta-\varepsilon}{\overset{\alpha_1}{\otimes}} \Rightarrow \underset{\varepsilon-\varepsilon_0}{\overset{\alpha_2}{\bigcirc}} \qquad A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}. \tag{3.20}$$

Its even generators fall into two sets of commutating angular momentum $\boldsymbol{J}$ and $\boldsymbol{L}$, $\boldsymbol{J}$ describing sp(2) and $\boldsymbol{L}$, o(3). The odd part constitutes the $(3, 2)$ representation of o(3)⊕sp(2), $V_{m,s}$, $m = +1, 0, -1$, and $s = \pm\frac{1}{2}$. They satisfy the following relations:

$$\begin{aligned} &[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \\ &[L_0, L_\pm] = \pm L_\pm \qquad [L_+, L_-] = 2L_0 \\ &[\boldsymbol{J}, \boldsymbol{L}] = 0 \end{aligned} \tag{3.21a}$$

$$\begin{aligned} &[L_0, V_{m,s}] = mV_{m,s} \qquad [J_0, V_{m,s}] = sV_{m,s} \\ &[L_\pm, V_{m,s}] = \sqrt{(1 \mp m)(1 \pm m + 1)}\, V_{m\pm1,s} \\ &[J_\pm, V_{m,s}] = \sqrt{(\tfrac{1}{2} \mp s)(\tfrac{1}{2} \pm s + 1)}\, V_{m,s\pm1} \end{aligned} \tag{3.21b}$$

$$\begin{aligned} &\{V_{1,\pm1/2}, V_{0,\mp1/2}\} = \mp 1/\sqrt{2}\, L_+ \\ &\{V_{-1,\mp1/2}, V_{0,\pm1/2}\} = \pm 1/\sqrt{2}\, L_- \\ &\{V_{1,\pm1/2}, V_{-1,\mp1/2}\} = \pm L_0 - 2J_0 \\ &\{V_{1,\pm1/2}, V_{-1,\pm1/2}\} = \pm 2J_\pm \\ &\{V_{0,\pm1/2}, V_{0,\pm1/2}\} = \mp 2J_\pm \\ &\{V_{0,1/2}, V_{0,-1/2}\} = 2J_0. \end{aligned} \tag{3.21c}$$

As before, generators $\boldsymbol{J}$ can be put into the oscillator form with the help of a pair of bosonic operators b and b^+:

$$J_+ = -\tfrac{1}{2}b^+b^+ \qquad 2J_0 = b^+b + \tfrac{1}{2} \qquad J_- = \tfrac{1}{2}bb. \tag{3.22}$$

The key point is how to deal the operators $\boldsymbol{L}$ and $V_{m,s}$. For this purpose let us consider the two-dimensional spinor space, on which the Clifford algebra is defined as

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB} \qquad A, B = 1, 2, 3. \tag{3.23}$$

It is well known that the matrices

$$M_{AB} = \frac{1}{4i}[\Gamma_A, \Gamma_B] \tag{3.24}$$

have the o(3) commutation relations and, furthermore, the Γs are a set of tensors transforming according to the three-dimensional vector representation of o(3):

$$[M_{AB}, \Gamma_C] = i(\delta_{AB}\Gamma_C - \delta_{BC}\Gamma_A). \tag{3.25}$$

In a recent paper [7] we have pointed out that with the help of a pair of fermionic operators a and a^+, $\{a, a^+\} = 1$, $a^2 = a^{+^2} = 0$, the combinations

$$\Gamma_1 = a + a^+ \qquad \Gamma_2 = i(a - a^+)$$

meet the requirement of the first two Γs:

$$\Gamma_1^2 = \Gamma_2^2 = 1 \qquad \{\Gamma_1, \Gamma_2\} = 0.$$

From the definition of the number operator $M = a^+ a$ one sees immediately that

$$Ma^+ = a^+(1-M) \qquad (1-M)a = aM \tag{3.26}$$

from which follows the properties of the operator $a_0 = (-1)^M$:

$$a_0^+ = a_0 \qquad a_0^2 = 1 \qquad a_0 a = -a a_0 \qquad a_0 a^+ = -a^+ a_0. \tag{3.27}$$

This means that a_0 can be chosen as Γ_3. Notice that the product

$$\Gamma_1\Gamma_2\Gamma_3 = i$$

is indeed proportional to the identity. The fact that the operator a_0, although being bosonic itself, anticommutates with fermionic operators a, a^+ and commutates with bosonic operators b, b^+, causes us to consider a_0 as an auxiliary fermionic operator with $a_0^+ = a_0$, $a_0^2 = 1$. Now having fixed the set of operators Γ_1, Γ_2 and Γ_3, one can construct the o(3) generators as in equations (3.23). By an appropriate diagonalization, one gets

$$L_+ = a^+ a_0 \qquad 2L_0 = 2M - 1 \qquad L_- = a_0 a. \tag{3.28}$$

And the set of operators $(a^+, a_0/\sqrt{2}, -a)$ forms the vector representation [7]. Therefore the combinations

$$\begin{aligned} &V_{1,+} = b^+ a^+ \qquad V_{0,+} = b^+ a_0/\sqrt{2} \qquad V_{-1,+} = -b^+ a \\ &V_{1,-} = b a^+ \qquad V_{0,-} = b a_0/\sqrt{2} \qquad V_{-1,-} = -ba \end{aligned} \tag{3.29}$$

do transform as the $(3, 2)$ representation under o(3)⊕sp(2). Equations (3.22), (3.28) and (3.29) complete the oscillator expression of Osp(3|2). It is not difficult to check that all the relations in equations (3.21) are completely satisfied.

In the standard notation, the generators corresponding to the simple roots can be chosen as follows:

$$\begin{aligned} &e_{\alpha_1} = b^+ a \qquad e_{-\alpha_1} = a^+ b \qquad h_1 = N + M \\ &e_{\alpha_2} = a^+ a_0 \qquad e_{-\alpha_2} = a_0 a \qquad h_2 = 2M - 1. \end{aligned} \tag{3.30}$$

Then the fundamental relations can be expressed as

$$\begin{aligned} &[h_i, e_{\alpha_j}] = a_{ij} e_{\alpha_j} \qquad [h_i, e_{-\alpha_j}] = -a_{ij} e_{-\alpha_j} \\ &\langle e_{\alpha_i}, e_{-\alpha_j} \rangle = \delta_{ij} h_j \end{aligned} \tag{3.31}$$

with a_{ij} being the Cartan matrix. Other generators can be obtained by suitable commutation, e.g. corresponding to the root $\alpha_1 + \alpha_2$ one has

$$[e_{\alpha_1}, e_{\alpha_2}] = [b^+ a, a^+ a_0] = b^+\{a, a^+\} a_0 = b^+ a_0.$$

Again, the quantum enveloping algebra of Osp(3|2) can be obtained simply by changing the oscillators a, a^+, b, b^+ in the step generators into their q-deformed counterparts $\tilde{a}$, $\tilde{a}^+$, $\tilde{b}$, $\tilde{b}^+$, as given in equation (3.13). Then

$$\{\tilde{e}_{\alpha_1}, \tilde{e}_{-\alpha_1}\} = \{\tilde{b}^+ \tilde{a}, \tilde{a}^+ \tilde{b}\} = [1+N][M] + [N][1-M] = [M+N] = [h_1] \tag{3.32a}$$

$$[\tilde{e}_{\alpha_2}, \tilde{e}_{-\alpha_2}] = [\tilde{a}^+ a_0, a_0 \tilde{a}] = [M] + [1-M] = [M - \tfrac{1}{2}]/[\tfrac{1}{2}] = [h_2]_{q^{1/2}}. \tag{3.32b}$$

The subscript $q^{1/2}$ indicates that the root α_2 is shorter than α_1.

The approach used here in this example can be generalized to high-rank superalgebra $B(m, n)$ without any difficulties. For $D(m, n)$ things become much simpler since the orthogonal subalgebra is defined on even-dimensional space, so that no parameter ε_0 (operator a_0) is needed.

4. The general case

The even part of $\mathrm{Osp}(M|2n)$, as mentioned above, is a direct sum of $\mathrm{o}(M)\oplus\mathrm{sp}(2n)$ and the odd part of $\mathrm{Osp}(M|2n)$ reduces for $M\neq 2$ to the $(M, 2n)$ representation of $\mathrm{o}(M)\oplus\mathrm{sp}(2n)$. In a recent paper [7], we have successfully put the symplectic algebra $\mathrm{sp}(2n)$ into the operator form by using the q-deformed bosonic oscillators. The same applies to the case of the orthogonal algebra $\mathrm{o}(M)$ by putting the Clifford algebra into the operator form and deforming fermionic oscillators in an appropriate way. Also given are the operator forms of the fundamental vector representations for both symplectic and orthogonal algebras. So we can now construct the BLS $\mathrm{Osp}(M|2n)$ in its entirety, with both fermionic and bosonic operators. As stressed above, special attention must be paid to the case for $M=2m+1$, where an auxiliary fermionic operator a_0 ($a_0^+=a_0$) is introduced [7] to simulate the effect of γ_5.

To realize the BLS $B(m,n)$, $B(0,n)$, $D(m,n)$ and $C(1+n)$ in a somewhat uniform way, we put the fermionic and bosonic operators together to set up a graded Clifford algebra C. The single fermion operator a_0, now denoted as c_0, will play the role of the 'centre' (commutating with all bosonic operators, while anticommutating with all fermionic ones) in the algebra C.

Algebra C is comprised as follows. For any orthosymplectic Lie superalgebra $\mathrm{Osp}(M|2n)$, to each bosonic parameter δ_k we introduce a pair of bosonic operators (b_k, b_k^+), and to each fermionic parameter ε_i, a pair of fermionic operators (a_i, a_i^+). (For B, ε_0 is assumed to associate with a_0, $a_0^+=a_0$, $a_0^2=1$.) Collecting $\{b_k\}$ and $\{a_i\}$ together, we get a set of graded operators $\{c_i\}$ and $\{c_i^+\}$, which satisfy

$$\begin{array}{llll} \langle c_i, c_j^+\rangle=\delta_{ij} & i,j\in I & & \\ \langle c_i, c_j\rangle=\langle c_i^+, c_j^+\rangle=0 & i,j\in I\cap[0] & & \\ c_0=c_0^+ \quad c_0^2=1 & \deg(c_0)=1 & I=I_0\cap I_1 & \end{array} \tag{4.1}$$

where $\langle a,b\rangle=ab-(-1)^{\deg(a)\deg(b)}ba$ for all $a,b\in C$ with $\deg(a)$, $\deg(b)$ being zero or 1 when their corresponding indices belong to I_0 or I_1; I_0 and I_1 are, thereafter, two specified sets of positive natural numbers. Algebra C is generated by 1, c_i and c_i^+.

It is then easy to check that the relations (1.9), with Cartan matrix (a_{ij}), given in equations (2.10), (2.11), (2.13) and (2.14), are reproduced by the following identifications:

(i) For $B(m,n)$

$$\begin{array}{ll} e_{\alpha_i}=c_i^+c_{i+1} & e_{-\alpha_i}=c_{i+1}^+c_i \\ h_i=(1+\delta_{i,n+m})c_i^+c_i-(-1)^{\deg(i)+\deg(i+1)}c_{i+1}^+c_{i+1} & i\in I \end{array} \tag{4.2}$$

with $I=I_0\cap I_1$, $I_0=[1,\ldots,n]$, $I_1=[n+1,\ldots,n+m]$ and $c_{n+m+1}=c_0$. Here the case for $m=0$ is included, i.e. $B(0,n)$.

(ii) For $D(m,n)$

$$\begin{array}{ll} e_{\alpha_i}=c_i^+c_{i+1} & e_{-\alpha_i}=c_{i+1}^+c_i \\ h_i=c_i^+c_i-(-1)^{\deg(i)+\deg(i+1)}c_{i+1}^+c_{i+1} & i\in I-[n+m] \\ e_{\alpha_{n+m}}=c_{n+m-1}^+c_{n+m}^+ & e_{-\alpha_{n+m}}=c_{n+m}c_{n+m-1} \\ h_{n+m}=c_{n+m-1}^+c_{n+m-1}+c_{n+m}^+c_{n+m}-1 & \end{array} \tag{4.3}$$

with $I=I_0\cap I_1$, $I_0=[1,\ldots,n]$, $I_1=[n+1,\ldots,n+m]$.

(iii) For $C(1+n)$

$$e_{\alpha_i}=c_i^+c_{i+1} \qquad e_{-\alpha_i}=c_{i+1}^+c_i \tag{4.4a}$$

$$h_i=c_i^+c_i-(-1)^{\deg(i)+\deg(i+1)}c_{i+1}^+c_{i+1} \qquad i\in I-[n+1] \tag{4.4b}$$

$$e_{\alpha_{n+1}}=-\frac{c_{n+1}^+c_{n+1}^+}{2} \qquad e_{-\alpha_{n+1}}=\frac{c_{n+1}c_{n+1}}{2} \tag{4.4c}$$

$$h_{n+1}=c_{n+1}^+c_{n+1}+\tfrac{1}{2} \tag{4.4d}$$

with $I=I_0\cap I_1$, $I_0=[2,\ldots,n+1]$, $I_1=[1]$.

Here we notice that the statistical property of the simple roots is automatically guaranteed by our construction, since for $X=ab$ we have $\deg(X)=\deg(a)+\deg(b)$ as used commonly.

5. *q*-deformation of BLS

A q-deformation of BLS is defined as follows:

$$\begin{aligned}&[h_i,h_j]=0\\ &[e_{\alpha_i},e_{-\alpha_j}]=\delta_{ij}[h_i]_{q_i}\\ &\langle h_i,e_{\pm\alpha_j}\rangle=\pm a_{ij}e_{\pm\alpha_j}\end{aligned} \tag{5.1}$$

where for given x, $[x]_q$ is defined as

$$[x]_q=\frac{q^x-q^{-x}}{q-q^{-1}} \tag{5.2}$$

and q is a quantum parameter.

In equations (5.1) the subscript q_i is defined as

$$q_i^2=q^{(\alpha_i,\alpha_i)_{\mathrm{E}}} \tag{5.3}$$

with $(\alpha_i,\alpha_i)_{\mathrm{E}}=(\alpha_i,\eta\alpha_i)$ called the Euclidean length of the roots α_i, i.e. for the inner product $(\,,)_{\mathrm{E}}$ we recover the Euclidean metric in the simple root system by sandwiching the η-matrix defined in section 2, which equals $+1$ for bosonic bases and -1 for fermionic bases:

$$\begin{aligned}&(\varepsilon_i,\varepsilon_j)_{\mathrm{E}}=\delta_{ij} \qquad (i,j=1,\ldots,m)\\ &(\varepsilon_0,\varepsilon_0)_{\mathrm{E}}=0 \qquad (\varepsilon_0,\varepsilon_j)_{\mathrm{E}}=0 \qquad (\varepsilon_0,\delta_k)_{\mathrm{E}}=0\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}&(\delta_k,\delta_l)_{\mathrm{E}}=\delta_{kl} \qquad (i,j=1,\ldots,m)\\ &(\varepsilon_i,\delta_k)_{\mathrm{E}}=0.\end{aligned} \tag{5.5}$$

We introduce q-deformed algebra C_q with a set of operators $\tilde{c}_i$, $\tilde{c}_i^+$ satisfying the following conditions:

$$\tilde{c}_i^+\tilde{c}_i=[N_i] \qquad \tilde{c}_i\tilde{c}_i^+=[1+(-1)^{\deg(i)}N_i] \qquad i\neq 0 \tag{5.6a}$$

$$[N_i,\tilde{c}_j^+]=\delta_{ij}\tilde{c}_i^+ \qquad [N_i,\tilde{c}_j]=-\delta_{ij}\tilde{c}_i \tag{5.6b}$$

$$\langle\tilde{c}_i,\tilde{c}_j^+\rangle=0 \qquad (i\neq j) \tag{5.6c}$$

$$\langle\tilde{c}_i,\tilde{c}_j\rangle=\langle\tilde{c}_i^+,\tilde{c}_j^+\rangle=0. \tag{5.6d}$$

A direct consequence of the definition is

$$(\tilde{c}_i^+)^2 = (\tilde{c}_i)^2 = 0 \qquad \text{for } \deg \tilde{c}_i = 1. \tag{5.7}$$

Now, as has been repeatedly used before [1, 4, 7], by keeping the generators in the Cartan subalgebra unaltered and replacing the operators c_i, c_i^+ in step generators by their deformed counterparts $\tilde{c}_i$, $\tilde{c}_i^+$, we go from classical BLSs to the corresponding quantum algebras. Here we only list the main results:

(i) $B(m, n)$

$$\begin{aligned}
&\tilde{e}_{\alpha_i} = \tilde{c}_i^+ \tilde{c}_{i+1} \qquad \tilde{e}_{-\alpha_i} = \tilde{c}_{i+1}^+ \tilde{c}_i \qquad (i = 1 \text{ to } n+m)\\
&h_i = N_i - (-1)^{\deg(i)+\deg(i+1)} N_{i+1}\\
&\tilde{e}_{\alpha_{n+m}} = \tilde{c}_{n+m}^+ c_0 \qquad \tilde{e}_{-\alpha_{n+m}} = c_0 \tilde{c}_{n+m}\\
&h_{n+m} = 2N_{n+m} - (-1)^{\deg(n+m)+1}.
\end{aligned} \tag{5.8}$$

(ii) $D(m, n)$

$$\begin{aligned}
&\tilde{e}_{\alpha_i} = \tilde{c}_i^+ \tilde{c}_{i+1} \qquad \tilde{e}_{-\alpha_i} = \tilde{c}_{i+1}^+ \tilde{c}_i \qquad (i = 1 \text{ to } n+m-1)\\
&h_i = N_i - (-1)^{\deg(i)+\deg(i+1)} N_{i+1}\\
&\tilde{e}_{\alpha_{n+m}} = \tilde{c}_{n+m-1}^+ \tilde{c}_{n+m}^+ \qquad \tilde{e}_{-\alpha_{n+m}} = \tilde{c}_{n+m} \tilde{c}_{n+m-1}\\
&h_{n+m} = N_{n+m-1} + N_{n+m} - 1.
\end{aligned} \tag{5.9}$$

(iii) $C(1+n)$

$$\begin{aligned}
&\tilde{e}_{\alpha_i} = \tilde{c}_i^+ \tilde{c}_{i+1} \qquad \tilde{e}_{-\alpha_i} = \tilde{c}_{i+1}^+ \tilde{c}_i \qquad (i = 1, \ldots, n)\\
&h_i = N_i - (-1)^{\deg(i)+\deg(i+1)} N_{i+1}\\
&\tilde{e}_{\alpha_{n+1}} = -\frac{1}{[2]} \tilde{c}_{n+1}^+ \tilde{c}_{n+1}^+ \qquad \tilde{e}_{-\alpha_{n+1}} = \frac{1}{[2]} \tilde{c}_{n+1} \tilde{c}_{n+1}\\
&h_{n+1} = N_{n+1} + \tfrac{1}{2}.
\end{aligned} \tag{5.10}$$

We now discuss the Serre relations [8]. We emphasize that in our approach, as illustrated in section 3, the whole set of generators of the algebras, including those of the non-simple roots, can be presented. Therefore, the closure of the algebras indeed implies the Serre relations. As is well known, the Serre relations for the q-deformed classical Lie algebras have been discussed by Jimbo [9] and can also be shown to be satisfied naturally in the oscillator representation. For instance, let us consider a case where two adjacent simple roots (which are both bosonic), say α_i and α_{i+1}, are connected by a single line in the Dynkin diagrams. The corresponding Serre relations read

$$\tilde{e}^2_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} - (q+q^{-1}) \tilde{e}_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} + \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} = 0. \tag{5.11}$$

In the oscillator representation (equations (5.8)–(5.10)) we have

$$\tilde{e}_{\alpha_i} = \tilde{c}_i^+ \tilde{c}_{i+1} \qquad \tilde{e}_{\alpha_{i+1}} = \tilde{c}_{i+1}^+ \tilde{c}_{i+2} \tag{5.12}$$

with the oscillators at i, $i+1$, $i+2$ having the same degree, i.e. all three oscillators are of the same kind, either bosonic or fermionic. As an illustration, we may take the bosonic one. Then we have

$$[\tilde{e}_{\alpha_i}, \tilde{e}_{\alpha_{i+1}}]_q \equiv \tilde{b}_i^+ \tilde{b}_{i+1} \tilde{b}_{i+1}^+ \tilde{b}_{i+2} - q \tilde{b}_{i+1}^+ \tilde{b}_{i+2} \tilde{b}_i^+ \tilde{b}_{i+1} = \tilde{b}_i^+ \tilde{b}_{i+1} q^{-N_{i+1}} \tag{5.13}$$

and then

$$[\tilde{e}_{\alpha_i}, [\tilde{e}_{\alpha_i}, \tilde{e}_{\alpha_{i+1}}]_q]_{q^{-1}} = \tilde{b}_i^+ \tilde{b}_{i+1} \tilde{b}_i^+ \tilde{b}_{i+1} q^{-N_{i+1}} - q^{-1} \tilde{b}_i^+ \tilde{b}_{i+1} q^{-N_{i+1}} \tilde{b}_i^+ \tilde{b}_{i+1} = 0 \tag{5.14}$$

which is, indeed, equation (5.11) with plus sign.

The feature for the Lie superalgebras is that, besides the case presented above, there exists a fermionic simple root in the Dynkin diagrams. In the case of one fermionic root simply connected with one bosonic root, things become a little complicated. However, we can show, by a straightforward calculation, that

$$\tilde{e}^2_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} - q^2 \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} = 0 \tag{5.15a}$$

if $\deg \alpha_i = 1$, $\deg \alpha_{i+1} = 0$, and

$$\tilde{e}^2_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} - (q + q^{-1}) \tilde{e}_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} + \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} = 0 \tag{5.15b}$$

if $\deg \alpha_i = 0$, $\deg \alpha_{i+1} = 1$.

All these relations can be considered as the super-form of the Serre relations similar to the ones given by Kulish *et al* [3].

As for the doubly-laced case, a similar consideration gives the following relations:

$$\tilde{e}^2_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} - (q + q^{-1}) \tilde{e}_{\pm\alpha_i} \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} + \tilde{e}_{\pm\alpha_{i+1}} \tilde{e}^2_{\pm\alpha_i} = 0 \tag{5.16a}$$

if $(\alpha_i, \alpha_i)_{\mathrm{E}} > (\alpha_{i+1}, \alpha_{i+1})_{\mathrm{E}}$ and

$$\tilde{e}^3_{\alpha_i} \tilde{e}_{\alpha_{i+1}} - (1 + q^2 + q^{-2})(\tilde{e}^2_{\alpha_i} \tilde{e}_{\alpha_{i+1}} \tilde{e}_{\alpha_i} - \tilde{e}_{\alpha_i} \tilde{e}_{\alpha_{i+1}} \tilde{e}^2_{\alpha_i}) - \tilde{e}_{\alpha_{i+1}} \tilde{e}^3_{\alpha_i} = 0 \tag{5.16b}$$

if $(\alpha_i, \alpha_i)_{\mathrm{E}} < (\alpha_{i+1}, \alpha_{i+1})_{\mathrm{E}}$.

Equations (5.11), (5.15) and (5.16) exhaust the possible connection of two adjacent simple roots in the Dynkin diagrams listed above in equations (2.10)–(2.14) respectively, and thus together with equations (5.8)–(5.10) complete the definitions of q-deformation of BLS $B(m, n)$, $B(0, n)$, $C(1+n)$ and $D(m, n)$.

The application of these quantum Lie superalgebras will be discussed elsewhere.

References

[1] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
Ng Y J 1990 *J. Phys. A: Math. Gen.* **23** 1023

[2] Hayashi T 1990 *Commun. Math. Phys.* **127** 129

[3] Kulish P P and Reshetikhin N Yu 1989 *Lett. Math. Phys.* **18** 143
Chaichian M and Kulish P P 1990 *Phys. Lett.* **234B** 72
Chaichian M, Kulish P P and Lukierski J 1990 *Phys. Lett.* **237B** 401
Deguchi T, Fujii A and Ito K 1990 *Phys. Lett.* **238B** 242

[4] Song X C 1990 *J. Phys. A: Math. Gen.* **23** L821

[5] Kač V 1977 *Adv. Math.* **26** 8; 1977 *Commun. Math. Phys.* **53** 31; 1977 *Lecture Notes in Mathematics* vol 676 (Berlin: Springer) p 596

[6] Frappat L, Sciarrino A and Sorba P 1989 *Commun. Math. Phys.* **121** 457
Han Q Z and Sun H Z 1983 *Commun. Theor. Phys.* **2** 1137

[7] Liao L and Song X C 1990 *Preprint* CCAST Beijing 90-31

[8] Serre J P 1987 *Complex Semisimple Lie Algebras* (Berlin: Springer)

[9] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247